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Stresses in rotating spheres grown by accretion

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Abstract

Analytical expressions are obtained for the stress field in a sphere that has grown by accretion while rotating about an axis at a speed which may vary during the accretion process. It is assumed that accretion occurs by the adherence of infinitesimal particles that are stress free at the instant of attachment and that the material of the sphere behaves elastically once it has accreted. The resulting stress field differs significantly from that predicted in a sphere that was 'manufactured' in a stress free state and then set to rotate. The implications of these differences are discussed in the context of the mechanisms for failure in accreted planetary bodies.

Keywords: Accretion; Elasticity; Spherical harmonics; Residual stresses; Asteroids; Comets; Planets; Rotational stresses

1. Introduction

The stress fields in planetary bodies influence many pertinent problems in planetary science such as tidal disruption (Dobrovolskis, 1990), impact scenarios (Asphaug et al., 2002), and equilibrium shape configurations for rubble piles (Washabaugh and Scheeres, 2002). The earliest solution of a problem of this class involving a solid was due to Chree (1895), who determined the stress field in an elastic self-gravitating ellipsoid spinning about a primary axis.

Chree's solution defines the stresses that would occur in a body that was somehow 'manufactured' in a stress free state, after which it is loaded by gravitational forces and caused to rotate. In practice, planetary objects such as comets and asteroids typically grow by accretion (Weidenschilling, 2000) and forces due to gravitation and rotation are present throughout the accretion process. There is every reason to believe that

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this will influence the final stress state in the body and that elastic removal of these forces would leave the body in a state of *residual stress* (Holsapple, 2001).

Brown and Goodman (1963) found the analytic solution for the stress field in an elastic self-gravitating spherical shell of inner radius r_1 and outer radius r_0 that had grown by accretion (they did not consider rotation). The corresponding solution for a solid sphere can be obtained by setting $r_1 = 0$. The resulting stresses are everywhere hydrostatic and there is a non-zero residual stress, in contrast to Chree's solution. In the present paper, we shall extend Brown and Goodman's results to allow for the effects of rotation, including the case where the rotational speed of the body changes in a fairly general fashion during the accretion process. The results show significant differences from those of Chree (1895). They therefore also imply that if the fully accreted sphere were then to be brought to rest (and hence unloaded), it would be left in a state of residual stress.

2. Effect of accretion on the stress field

We consider a spherical planetary object that grows by accretion from zero up to a maximum radius a, while rotating at speed Ω, which may vary during the accretion process. We assume that the accreting particles are much smaller than the growing sphere so that the latter can at all times be treated as a continuum and the instantaneous radius of the sphere treated as a continuous, monotonically increasing function $s(t)$ of time t. Because of this assumption, the results obtained are only valid for planetary bodies such as asteroids and comets that were grown by the accumulation of much smaller particles (Weidenschilling, 2000); they cannot be used for the analysis of the major planets, since these were formed by the accumulation of a relatively small number of planetesimals.

We assume that the material of the sphere behaves elastically *once it has accreted*. In other words, the deformation of a particle at radius R will be elastic for $t > \tau(R)$, where $\tau(R)$ defines the time[1] at which the radius of the sphere has just reached R. The equations of elasticity have no meaning for $t < \tau(R)$, since in this time range there is no material at radius R.

It follows that the *change* in the stress field between times t_1 and t_2 ($t_2 > t_1 > \tau(R)$) must satisfy the elastic compatibility equations and in particular that the time derivative of the stress field $\dot{\boldsymbol{\sigma}}$ satisfies these equations at radius R. If we now integrate $\dot{\boldsymbol{\sigma}}$ with respect to time t, we conclude that the most general stress field in an accreting elastic body can be written in the form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^o + \boldsymbol{\sigma}^c(t), \tag{1}$$

where $\boldsymbol{\sigma}^c(t)$ is a time-varying, elastic (compatible) stress field and $\boldsymbol{\sigma}^o$ *represents an arbitrary function of integration (in this case an arbitrary function of radius R, but not a function of t)*. The decomposition of $\boldsymbol{\sigma}$ into compatible and time-independent components is not unique, but one such decomposition would be to define $\boldsymbol{\sigma}^o$ as the residual stress that would remain *after complete accretion* if the forces due to gravitation, rotation and any boundary tractions were removed.

2.1. Accretion boundary conditions

The solution due to Chree (1895) is based on the assumption that there is no residual stress ($\boldsymbol{\sigma}^o = 0$), in which case the conventional traction-free boundary condition at $R = a$ is sufficient to determine $\boldsymbol{\sigma}^c$. In the case of accretion, additional boundary conditions are needed to determine $\boldsymbol{\sigma}^o$ and these must be based on the nature of the accretion process.

[1] Notice that τ is the function inverse to s—i.e. $s\{\tau(R)\} = R$.

If the sphere were to grow by the instantaneous addition of a thin spherical shell of inside radius s, it would be at least conceivable that this shell could come into being in a state of 'membrane' stress.[2] However, the actual accretion process involves the successive adherence of many small particles to the solid sphere and except for the actual area of contact, the surface of these particles is traction-free at the instant of adherence. We conclude that the surface of the accreting sphere is not merely traction-free, but completely stress-free. In other words, all six stress components must be zero at $R = s(t)$. This provides sufficient conditions to determine the complete stress field $\boldsymbol{\sigma}$. Notice that this situation is analogous to that occurring in the solidification of castings, where it is assumed that no instantaneous change in stress state occurs as a particle changes from the liquid (and hence hydrostatic) state to the solid state (Richmond and Tien, 1971).

3. Stress field due to rotation alone

The problem as stated is linear and hence the resulting stresses are the sum of those due to gravitational forces and rotational inertia forces respectively, considered in isolation. The stress field in an accreted sphere due to pure gravitational loading has already been given by Brown and Goodman (1963), so in this section we consider the problem in which the only loading is the inertia force field $\boldsymbol{f}$ due to rotation at speed Ω, given by

$$f_r = \rho\Omega^2 r; \quad f_\theta = f_z = 0 \tag{2}$$

in cylindrical polar coordinates r, θ, z. This is conveniently defined in the form

$$\boldsymbol{f} = -\nabla V, \tag{3}$$

where the body force potential

$$V = -\frac{1}{2}\rho r^2\Omega^2 \tag{4}$$

or in spherical polar coordinates R, θ, β

$$V = -\frac{1}{2}\rho R^2\Omega^2\sin^2\beta. \tag{5}$$

The rotational speed might change during the accretion process and hence is a function of t. Time enters the problem only as a parameter as long as the angular acceleration is not sufficiently large as to cause significant additional inertia forces. However, any time-dependence of Ω will also imply a dependence on the instantaneous radius $s(t)$ and this will influence the final stress field through the stress-free accretion boundary condition.

In view of the possible time-dependence of the loading, the body force must be accounted for in the solution for the time-varying term $\boldsymbol{\sigma}^c(t)$. It then follows that the residual stress term $\boldsymbol{\sigma}^o$ must satisfy the equilibrium equations in the absence of body force.

3.1. The time-varying stress field $\boldsymbol{\sigma}^c(t)$

A sufficiently general solution for $\boldsymbol{\sigma}^c(t)$ can be written in terms of Green and Zerna's solution A and B and a body force potential in the form

[2] This occurs if, for example, a long thin plate is tightly wound around a drum (Yagoda, 1980; Debesis and Burns, 2003).

$$\sigma^c_{RR} = \frac{vV}{1-v} + \frac{\partial^2\phi}{\partial R^2} + R\cos\beta\frac{\partial^2\omega}{\partial R^2} - 2(1-v)\frac{\partial\omega}{\partial R}\cos\beta + \frac{2v}{R}\frac{\partial\omega}{\partial\beta}\sin\beta \tag{6}$$

$$\sigma^c_{\theta\theta} = \frac{vV}{1-v} + \frac{1}{R}\frac{\partial\phi}{\partial R} + \frac{\cot\beta}{R^2}\frac{\partial\phi}{\partial\beta} + \frac{\cos^2\beta}{R\sin\beta}\frac{\partial\omega}{\partial\beta} + (1-2v)\frac{\partial\omega}{\partial R}\cos\beta + \frac{2v}{R}\frac{\partial\omega}{\partial\beta}\sin\beta \tag{7}$$

$$\sigma^c_{\beta\beta} = \frac{vV}{1-v} + \frac{1}{R}\frac{\partial\phi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\phi}{\partial\beta^2} + \frac{\cos\beta}{R}\frac{\partial^2\omega}{\partial\beta^2} + (1-2v)\frac{\partial\omega}{\partial R}\cos\beta + \frac{2(1-v)}{R}\frac{\partial\omega}{\partial\beta}\sin\beta \tag{8}$$

$$\sigma^c_{\beta R} = \frac{1}{R}\frac{\partial^2\phi}{\partial\beta\partial R} - \frac{1}{R^2}\frac{\partial\phi}{\partial\beta} + \cos\beta\frac{\partial^2\omega}{\partial\beta\partial R} + (1-2v)\frac{\partial\omega}{\partial R}\sin\beta - \frac{2(1-v)}{R}\frac{\partial\omega}{\partial\beta}\cos\beta \tag{9}$$

$$\sigma^c_{\theta R} = \sigma^c_{\theta\beta} = 0, \tag{10}$$

where ω and ϕ are two time-dependent potential functions satisfying

$$\nabla^2\omega = 0 \tag{11}$$

$$\nabla^2\phi = \frac{(1-2v)V}{(1-v)} = -\frac{\rho(1-2v)}{2(1-v)}R^2\Omega^2(s)\sin^2\beta, \tag{12}$$

from Barber (2002), Table 19.2 and Section 18.5.1.

It is easily verified that the particular solution

$$\phi_{\mathrm{P}} = \frac{\rho R^4\Omega^2(s)(1-2v)(5\cos 2\beta - 3)}{280(1-v)} \tag{13}$$

satisfies (12) and hence the general solution of this equation can be written as the sum of ϕ_{P} and an arbitrary harmonic function ϕ_{H}. The harmonic functions ω, ϕ_{H} can be written in terms of spherical harmonics with time-dependent coefficients. They must be chosen to enable homogeneous boundary conditions to be satisfied at $R = s$ and hence we can be guided in the choice of appropriate spherical harmonics by the terms contributed by ϕ_{P} and V in Eqs. (6)–(10). We write

$$\begin{aligned}\phi &= \phi_{\mathrm{P}} + \phi_{\mathrm{H}}\\ &= \frac{\rho R^4\Omega^2(s)(1-2v)(5\cos 2\beta - 3)}{280(1-v)} + \frac{A_1R^2(3\cos 2\beta + 1)}{4} + \frac{A_2R^4(35\cos 4\beta + 20\cos 2\beta + 9)}{64}\end{aligned} \tag{14}$$

$$\omega = B_1R\cos\beta + \frac{B_2R^3(5\cos 3\beta + 3\cos\beta)}{8}, \tag{15}$$

where A_1, A_2, B_1, B_2 will generally depend parametrically upon time t and hence also on the instantaneous radius s of the partially accreted sphere.

Substituting (14), (15) and (5) into (6)–(10) we obtain the corresponding stress field as

$$\begin{aligned}\sigma^c_{RR} &= \frac{A_1}{2} - B_1 + \frac{3(9A_2 + 4vB_2)R^2}{16} - \frac{\rho\Omega^2(s)(18-v)R^2}{140(1-v)}\\ &\quad + \left[\frac{3A_1}{2} - B_1(1-2v) + \frac{3(5A_2 + 2B_2v)R^2}{4} + \frac{\rho\Omega^2(s)(6-5v)R^2}{28(1-v)}\right]\cos 2\beta\\ &\quad + \frac{15}{16}(7A_2 + 4B_2v)R^2\cos 4\beta\end{aligned} \tag{16}$$

$$\sigma^c_{\theta\theta} = -A_1 - 2B_1\nu - \frac{3[3A_2 + 2B_2(1+\nu)]R^2}{4} - \frac{\rho\Omega^2(s)(11+13\nu)R^2}{140(1-\nu)} - \left\{\frac{3[5A_2 + 2B_2(1+3\nu)]}{4} - \frac{\rho\Omega^2(s)(1+5\nu)}{28(1-\nu)}\right\}R^2\cos 2\beta \tag{17}$$

$$\sigma^c_{\beta\beta} = \frac{A_1}{2} - B_1 + \frac{3(3A_2 - 4B_2\nu)R^2}{16} - \frac{\rho\Omega^2(s)(6+23\nu)R^2}{140(1-\nu)} - \left[\frac{3A_1 - 2B_1(1-2\nu) + 3B_2(2+\nu)R^2}{2} - \frac{\nu\rho\Omega^2(s)R^2}{4(1-\nu)}\right]\cos 2\beta - \frac{15}{16}(7A_2 + 4B_2\nu)R^2\cos 4\beta \tag{18}$$

$$\sigma^c_{\beta R} = -\left\{\frac{3A_1}{2} - B_1(1-2\nu) + \frac{3[5A_2 + 4B_2(1+\nu)]R^2}{8} + \frac{3\rho\Omega^2(s)(1-2\nu)R^2}{28(1-\nu)}\right\}\sin 2\beta - \frac{15}{16}(7A_2 + 4B_2\nu)R^2\sin 4\beta \tag{19}$$

$$\sigma^c_{\theta R} = \sigma^c_{\theta\beta} = 0. \tag{20}$$

Eqs. (16)–(20) define a sufficiently general description of the time-varying compatible stress field $\boldsymbol{\sigma}^c(t)$.

3.2. The residual stress field

The residual stress $\boldsymbol{\sigma}^o$ is not required to satisfy the equations of compatibility, but it is independent of time and must satisfy the equilibrium equations

$$\frac{\partial\sigma_{RR}}{\partial R} + \frac{1}{R}\frac{\partial\sigma_{\beta R}}{\partial\beta} + \frac{\sigma_{\beta R}\cot\beta}{R} + \frac{(2\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\beta\beta})}{R} = 0 \tag{21}$$

$$\frac{\partial\sigma_{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial\sigma_{\beta\theta}}{\partial\beta} + \frac{1}{R\sin\beta}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{3\sigma_{R\theta}}{R} + \frac{2\sigma_{\beta\theta}\cot\beta}{R} = 0 \tag{22}$$

$$\frac{\partial\sigma_{R\beta}}{\partial R} + \frac{1}{R}\frac{\partial\sigma_{\beta\beta}}{\partial\beta} + \frac{3\sigma_{R\beta}}{R} + \frac{(\sigma_{\beta\beta} - \sigma_{\theta\theta})\cot\beta}{R} = 0. \tag{23}$$

(Saada, 1974, Section 7.12). The second equilibrium equation (22) is identically satisfied in view of the axisymmetry of the problem.

By analogy with Eqs. (16)–(20), a suitable general form[3] is

$$\sigma^o_{RR} = g_1(R) + g_2(R)\cos 2\beta \tag{24}$$

$$\sigma^o_{\theta\theta} = g_3(R) + g_4(R)\cos 2\beta \tag{25}$$

$$\sigma^o_{\beta\beta} = g_5(R) + g_6(R)\cos 2\beta \tag{26}$$

[3] At first sight, it seems that we also would need terms varying with $\cos(4\beta)$, $\sin(4\beta)$ as in Eqs. (16)–(20), but these terms are eliminated by the stress-free boundary conditions and are therefore omitted here in the interests of brevity.

$$\sigma^o_{\beta R} = g_7(R)\sin 2\beta \tag{27}$$

$$\sigma^o_{\theta R} = \sigma^o_{\theta\beta} = 0, \tag{28}$$

where $g_1,\ldots,g_7$ are arbitrary functions of R.

Substituting (24)–(28) into the equilibrium equations (21) and (23) and equating coefficients of the Fourier terms in β yields the four ordinary differential equations

$$Rg_2'(R) + 2g_2(R) - g_4(R) - g_6(R) + 3g_7(R) = 0 \tag{29}$$

$$Rg_1'(R) + 2g_1(R) - g_3(R) - g_5(R) + g_7(R) = 0 \tag{30}$$

$$Rg_7'(R) - 2g_3(R) - g_4(R) + 2g_5(R) - g_6(R) + 3g_7(R) = 0 \tag{31}$$

$$-Rg_7'(R) - g_4(R) + 3g_6(R) - 3g_7(R) = 0 \tag{32}$$

and these can be used to eliminate g_3, g_4, g_5, g_6 in Eqs. (24)–(28). Thus, the most general stress field of the form (24)–(28) that satisfies the equilibrium equations can be written in terms of three unknown functions of R as

$$\sigma^o_{RR} = g_1(R) + g_2(R)\cos 2\beta \tag{33}$$

$$\sigma^o_{\theta\theta} = g_1(R) - \frac{1}{2}\left(g_2(R) - g_7(R) - Rg_1'(R)\right) - \frac{R}{4}\left(g_2'(R) - g_7'(R)\right) + \frac{1}{4}\left[6(g_2(R) + g_7(R)) + R(3g_2'(R) - g_7'(R))\right]\cos 2\beta \tag{34}$$

$$\sigma^o_{\beta\beta} = g_1(R) + \frac{1}{2}\left(g_2(R) + g_7(R) + Rg_1'(R)\right) + \frac{R}{4}\left(g_2'(R) - g_7'(R)\right) + \frac{1}{4}\left[2g_2(R) + 6g_7(R) + R(g_2'(R) + g_7'(R))\right]\cos 2\beta \tag{35}$$

$$\sigma^o_{R\beta} = g_7(R)\sin 2\beta \tag{36}$$

$$\sigma^o_{\theta R} = \sigma^o_{\theta\beta} = 0. \tag{37}$$

The full stress field is then obtained by combining the corresponding stress components from Eqs. (16)–(20) and (33)–(37).

3.3. Stress free boundary conditions

All six stress components must be zero at the instantaneous boundary of the sphere $R = s(t)$ for all values of β. Setting the coefficients of the Fourier terms $\cos 4\beta$, $\cos 2\beta$ etc. to zero in each of the resulting equations yields

$$15(7A_2 + 4B_2 v)s^2 = 0 \tag{38}$$

$$7(1-v)[6A_1 - 4B_1(1-2v) + 3(5A_2 + 2B_2 v)s^2] + (6-5v)\rho\Omega^2(s)s^2 + 28(1-v)g_2(s) = 0 \tag{39}$$

$$35(1-v)[8(A_1 - 2B_1) + 3(9A_2 + 4B_2 v)s^2] - 4(18-v)\rho\Omega^2(s)s^2 + 560(1-v)g_1(s) = 0 \tag{40}$$

$$\{21(1-\nu)[5A_2+2B_2(1+3\nu)]-(1+5\nu)\rho\Omega^2(s)\}s^2-7(1-\nu)\left[6(g_2(s)+g_7(s))+s(3g_2'(s)-g_7'(s))\right]=0 \tag{41}$$

$$\begin{aligned}&35(1-\nu)\{4(A_1+2B_1\nu)+3[3A_2+2B_2(1+\nu)]s^2\}+(11+13\nu)\rho\Omega^2(s)s^2\\&\quad-35(1-\nu)\left[2(2g_1(s)-g_2(s)+g_7(s))+s(2g_1'(s)-g_2'(s)+g_7'(s))\right]=0\end{aligned} \tag{42}$$

$$15(7A_2+4B_2\nu)s^2=0 \tag{43}$$

$$\begin{aligned}&2(1-\nu)[3A_1+3B_2(2+\nu)s^2-2B_1(1-2\nu)]-\nu\rho\Omega^2(s)s^2\\&\quad-(1-\nu)\left[2g_2(s)+6g_7(s)+s(g_2'(s)+g_7'(s))\right]=0\end{aligned} \tag{44}$$

$$\begin{aligned}&35(1-\nu)[8(A_1-2B_1)+3(3A_2-4B_2\nu)s^2]-4(6+23\nu)\rho\Omega^2(s)s^2\\&\quad+140(1-\nu)\left[2(2g_1(s)+g_2(s)+g_7(s))+s(2g_1'(s)+g_2'(s)-g_7'(s))\right]=0\end{aligned} \tag{45}$$

$$15(7A_2+4B_2\nu)s^2=0 \tag{46}$$

$$7(1-\nu)\{4[3A_1-2B_1(1-2\nu)]+3[5A_2+4B_2(1+\nu)]s^2\}+6(1-2\nu)\rho\Omega^2(s)s^2-56(1-\nu)g_7(s)=0 \tag{47}$$

where we recall that A_1, A_2, B_1, B_2 and Ω are also functions of s.

Only seven of the simultaneous ODEs (38)–(47) are linearly independent and they can be solved for the seven functions A_1, A_2, B_1, B_2, g_1, g_2, g_7. The solution contains three arbitrary constants of integration, but these prove to have no effect on the final stress field which is obtained as

$$\begin{aligned}\sigma_{RR}=&\frac{\rho}{10(7+5\nu)(1-\nu)}\left[(17-24\nu-25\nu^2)\int_R^a\Omega^2(s)s\,\mathrm{d}s+(9+7\nu)\Omega^2(a)(a^2-R^2)\right]\\&-\frac{\rho}{2(7+5\nu)}\left[(1+\nu)\int_R^a\Omega^2(s)s\,\mathrm{d}s+(3+2\nu)\Omega^2(a)(a^2-R^2)\right]\cos 2\beta\end{aligned} \tag{48}$$

$$\begin{aligned}\sigma_{\theta\theta}=&\frac{\rho}{10(7+5\nu)(1-\nu)}\left[2(11-12\nu-15\nu^2)\int_R^a\Omega^2(s)s\,\mathrm{d}s+2(12+\nu-5\nu^2)\Omega^2(a)(a^2-R^2)\right.\\&\left.+(11-12\nu-15\nu^2)R^2(\Omega^2(a)-\Omega^2(R))\right]-\frac{\rho(1+\nu)R^2(\Omega^2(a)-\Omega^2(R))\cos(2\beta)}{2(7+5\nu)}\end{aligned} \tag{49}$$

$$\begin{aligned}\sigma_{\beta\beta}=&\frac{\rho}{10(7+5\nu)(1-\nu)}\left[(17-24\nu-25\nu^2)\int_R^a\Omega^2(s)s\,\mathrm{d}s+2(3-6\nu-5\nu^2)R^2(\Omega^2(a)-\Omega^2(R))\right.\\&\left.+(9+7\nu)\Omega^2(a)(a^2-R^2)\right]+\frac{\rho\cos(2\beta)}{2(7+5\nu)}\left[(1+\nu)\int_R^a\Omega^2(s)s\,\mathrm{d}s+(3+2\nu)\Omega^2(a)(a^2-R^2)\right]\end{aligned} \tag{50}$$

$$\sigma_{\beta R}=\frac{\rho\sin(2\beta)}{2(7+5\nu)}\left[(1+\nu)\int_R^a\Omega^2(s)s\,\mathrm{d}s+(3+2\nu)\Omega^2(a)(a^2-R^2)\right] \tag{51}$$

$$\sigma_{\theta R}=\sigma_{\theta\beta}=0. \tag{52}$$

These equations define the state of stress in a sphere that has grown by accretion while rotating about its own axis. The effect of self-gravitation can be included by superposing the stress components

$$\sigma_{RR} = \sigma_{\theta\theta} = \sigma_{\beta\beta} = -\frac{2\pi\rho^2 G(a^2 - R^2)}{3}; \quad \sigma_{\beta R} = \sigma_{\theta R} = \sigma_{\theta\beta} = 0, \tag{53}$$

from Brown and Goodman (1963), where G is the universal gravitational constant.[4]

4. Failure mechanisms in accreted planetary bodies

Dones and Tremaine (1993) discuss the problem in which a cluster of particles orbiting a massive central body accretes into a single orbiting body. In particular, they conclude that if the statistical dispersion of the accreting particles is sufficiently small and/or the attraction to the central body is relatively weak, the rotational speed Ω will remain approximately constant during accretion. At the other extreme, if gravitational attraction is strong and dispersion is large, the rotational speed increases during accretion with the square of the instantaneous radius—i.e. $\Omega(s) \sim s^2$.

4.1. Constant rotational speed

If the rotational speed remains constant during accretion, Eqs. (48)–(52) reduce to the simple form

$$\sigma_{RR} = \frac{1}{4}\rho\Omega^2(a^2 - R^2)(1 - \cos 2\beta) - \frac{2\pi}{3}\rho^2 G(a^2 - R^2) \tag{54}$$

$$\sigma_{\theta\theta} = \frac{1}{2}\rho\Omega^2(a^2 - R^2) - \frac{2\pi}{3}\rho^2 G(a^2 - R^2) \tag{55}$$

$$\sigma_{\beta\beta} = \frac{1}{4}\rho\Omega^2(a^2 - R^2)(1 + \cos 2\beta) - \frac{2\pi}{3}\rho^2 G(a^2 - R^2) \tag{56}$$

$$\sigma_{\beta R} = \frac{1}{4}\rho\Omega^2(a^2 - R^2)\sin 2\beta \tag{57}$$

$$\sigma_{\theta R} = \sigma_{\theta\beta} = 0. \tag{58}$$

where we have included the self-gravitational stresses from Eq. (53).

Notice that both the gravitational and rotational terms in these equations vary with $(a^2 - R^2)$ and are independent of Poisson's ratio ν. In fact, the three principal stresses prove to be independent of polar angle β, being

$$\sigma_1 = \sigma_2 = \left(\frac{1}{2}\rho\Omega^2 - \frac{2\pi}{3}\rho^2 G\right)(a^2 - R^2); \quad \sigma_3 = -\frac{2\pi}{3}\rho^2 G(a^2 - R^2), \tag{59}$$

where σ_1, σ_2 act in the equatorial plane, while σ_3 acts in the polar direction. The principal stresses will be everywhere compressive if

$$\Lambda \equiv \frac{\Omega^2}{\pi\rho G} < \frac{4}{3} = \Lambda_P. \tag{60}$$

[4] Alternatively, we can write $\sigma_{RR} = \sigma_{\theta\theta} = \sigma_{\beta\beta} = -\rho g(a^2 - R^2)/2a$, where g is the value of the gravitational acceleration at the surface of the accreted sphere.

For $\Lambda > \Lambda_P$, σ_1, σ_2 become tensile, while σ_3 remains compressive. Notice however that the component of local gravitational acceleration normal to the instantaneous surface of the sphere (modified by rotation) is

$$g = \left(\frac{4}{3} - \Lambda \sin^2\beta\right)\pi\rho G s \tag{61}$$

and hence $\Lambda > \Lambda_P$ corresponds to a condition in which a particle placed near the equator would be thrown off into space. Accretion under these conditions could occur only if impacting particles were to become adhered to the surface as a result of the impact process with some non-zero value of cohesive strength. The value of g at the poles remains positive for all Λ, suggesting that the assumption of uniform accretion for all β is unrealistic for large Λ.

Table 1 lists estimated values of Λ for a range of planetary objects (Cox, 2000; Hilton, 2002; Mitchell et al., 1996; Pravec et al., 2000; IAU, 2003). For the major planets, the inequality (60) is clearly satisfied, as it is for most known asteroids. Some exceptions are listed in the fourth group of Table 1, but these are believed to be primarily monolithic fragments of larger bodies resulting from collisions (Pravec et al., 2000). The asteroids in the second group (Ceres, Pallas and Vesta) are closest to meeting the conditions assumed in this analysis, in that they are approximately spherical and are believed to have resulted from accretion of a large number of smaller particles (Weidenschilling, 2000). It is also believed that their present spin rate is close to that during the accretion process (Davis et al., 1988).

The data given in Table 1 show that tensile stresses are unlikely to arise in many planetary objects because of the dominance of the compressive stresses due to self-gravitation. However, the stresses due to gravitation are everywhere hydrostatic (see Eq. (53)) and hence cannot cause failure associated with shear deformation and governed (for example) by the Mises, Tresca or Mohr–Coulomb failure critera. The Mises stress (equivalent uniaxial tensile stress)

$$\begin{aligned}\sigma_E &= \sqrt{\sigma_{RR}^2 + \sigma_{\theta\theta}^2 + \sigma_{\beta\beta}^2 - \sigma_{RR}\sigma_{\theta\theta} - \sigma_{\theta\theta}\sigma_{\beta\beta} - \sigma_{\beta\beta}\sigma_{RR} + 3\sigma_{R\theta}^2 + 3\sigma_{\theta\beta}^2 + 3\sigma_{\beta R}^2} \\ &= \frac{1}{2}\rho\Omega^2(a^2 - R^2),\end{aligned} \tag{62}$$

Table 1
The dimensionless ratio Λ for some planets (group 1) and asteroids (groups 2–4)

Group	Object	Semi-axis ratios		Density, (ρ) kg/m^3	Rotational speed, (Ω) rad/s	Λ, $\Omega^2/(\pi\rho G)$
		a/b	b/c			
1	Earth	1.000	1.003	5515	7.30×10^{-5}	0.00461
	Mars	1.000	1.009	3940	7.09×10^{-5}	0.00609
	Jupiter	1.000	1.063	1330	1.76×10^{-4}	0.111
2	1 Ceres	1.00	1.06	2060 ± 50	1.92×10^{-4}	0.0834–0.0875
	2 Pallas	1.10	1.05	2900 ± 300	2.23×10^{-4}	0.0742–0.0914
	4 Vesta	1.07	1.20	3500 ± 200	3.27×10^{-4}	0.138–0.155
3	16 Psyche	1.29	1.31	1800 ± 600	4.16×10^{-4}	0.347–0.688
	45 Eugenia	1.36	1.45	$1200+600, -300$	3.06×10^{-4}	0.248–0.497
	87 Sylvia	1.42	1.26	1600 ± 300	3.37×10^{-4}	0.285–0.417
4	1998 WB_2	*	*	*	5.58×10^{-3}	36.9–149
	1999 TY_2	*	*	*	1.44×10^{-2}	247–987
	2000 WL_{10}	*	*	*	1.09×10^{-2}	142–567

Notice that data in group 1 is included only for the purpose of providing some perspective for the numerical values of the spin parameter $\Lambda = \Omega^2/\rho G$, since the accretion assumption of the present analysis does not accurately represent the growth process of the major planets. The parameters a, b, and c are the semi-axes of an approximating ellipsoid such that $a \geqslant b \geqslant c$. No reliable information is available for the parameters marked *. The range of values of Λ quoted for these objects was based on the assumption of a density in the range $1000 < \rho < 4000$ kg/m^3.

while the maximum shear stress is

$$\tau_{\max} = \frac{\rho\Omega^2(a^2 - R^2)}{4}. \tag{63}$$

Both these expressions reach a maximum value at the center of the sphere. However, the Mohr–Coulomb criterion, which is arguably the most appropriate for an accreted body, depends on the maximum value of the ratio of the shear stress to the compressive normal stress on a given plane. This ratio is found to be

$$\left(\frac{\tau}{-\sigma}\right)_{\max} = \left[\frac{8}{3\Lambda}\left(\frac{8}{3\Lambda} - 2\right)\right]^{-1/2} \tag{64}$$

and it is independent of R, β. In other words, all points in the sphere are in the same condition relative to the Mohr–Coulomb failure criterion. Equating this expression to a critical coefficient of friction f for slip to occur on an internal plane, we find that all points in the accreting sphere will remain within the failure envelope as long as

$$\Lambda < \frac{8}{3}\left(\frac{\sin\phi}{1 + \sin\phi}\right) = \Lambda_{\text{M–C}}, \tag{65}$$

where $\phi = \arctan(f)$ is the angle of friction. The Mohr–Coulomb falure parameter $\Lambda_{\text{M–C}}$ increases monotonically with the coefficient of friction from $\Lambda_{\text{M–C}} = 0$ at $f = 0$ to the limit $\Lambda_{\text{M–C}} \to \Lambda_{\text{P}}$ as $f \to \infty$.

4.2. Comparison with Chree's solution

Chree's solution for the stress field due to rotation and self-gravitation, based on the assumption of zero residual stress, is[5]

$$\sigma_{RR} = \frac{\rho\Omega^2(a^2 - R^2)[9 + 7v - 5(3 + 2v)(1 - v)\cos 2\beta]}{10(7 + 5v)(1 - v)} - \frac{2\pi\rho^2 G(3 - v)(a^2 - R^2)}{15(1 - v)} \tag{66}$$

$$\sigma_{\theta\theta} = \frac{\rho\Omega^2[2(12 + v - 5v^2)a^2 - (13 + 14v + 5v^2)R^2 - 5(1 - v^2)R^2\cos 2\beta]}{10(7 + 5v)(1 - v)} - \frac{2\pi\rho^2 G[(3 - v)a^2 - (1 + 3v)R^2]}{15(1 - v)} \tag{67}$$

$$\sigma_{\beta\beta} = \frac{\rho\Omega^2[5(3 + 2v)(1 - v)(a^2 - R^2)\cos 2\beta + (9 + 7v)a^2 - (3 + 19v + 10v^2)R^2]}{10(7 + 5v)(1 - v)} - \frac{2\pi\rho^2 G[(3 - v)a^2 - (1 + 3v)R^2]}{15(1 - v)} \tag{68}$$

$$\sigma_{\beta R} = \frac{(3 + 2v)\rho\Omega^2(a^2 - R^2)\sin 2\beta}{2(7 + 5v)} \tag{69}$$

$$\sigma_{\theta R} = \sigma_{\theta\beta} = 0. \tag{70}$$

[5] One way to obtain the rotational terms in these equations is to set $\Omega(s) = 0$ for $0 < s < a$ and $\Omega(a) = \Omega$ in Eqs. (54)–(58). In other words, to consider the special case where the sphere is not allowed to rotate until it is fully accreted.

Clearly this stress field satisfies traction-free boundary conditions on the sphere's surface, but not the stress-free conditions required by the accretion solution. The principal stresses are everywhere compressive if

$$\Lambda < \frac{2(7+5\nu)(1-2\nu)}{3(4-3\nu+5\nu^2)} = \Lambda_{\mathrm{P}}^{(C)}. \tag{71}$$

However, in contrast to the accretion solution (60), this criterion is strongly dependent on Poisson's ratio. In particular, if the material is incompressible ($\nu = 0.5$) the hoop stress $\sigma_{\theta\theta}$ at the equator will be tensile for all values of Λ, implying the possibility of tensile failure due to rotation if the material has little cohesive strength.

For $\nu = 0.5$, the stresses in Chree's solution resulting from gravitational loading alone are everywhere hydrostatic and identical to the accretion solution of Brown and Goodman (1963) given in Eq. (53). For all other values of ν (compressible materials), the gravitational stresses are not hydrostatic (except at the sphere's center), implying that gravitational loading will contribute significantly to the Mises stress and the maximum shear stress. In fact, for the bodies in the first three groups in Table 1, these stresses are dominated by gravitational effects unless Poisson's ratio is close to 0.5, and the rotational loading actually reduces the maximum value, which therefore occurs at the poles where rotational stresses are a minimum.

This comparison shows that the stress field in a rotating body without residual stress is qualitatively different from that in a body that grows by accretion while rotating. In fact, the residual stress for the accreted body could be found by subtracting Chree's solution from the accreted solution of Eqs. (54)–(58).

As a particular example, Fig. 1 shows a contour plot of the Mises stress for the asteroid Ceres. The resulting stress fields are completely different. In Chree's solution (a), the maximum stress occurs at the poles and is 29.2 MPa and there is a minimum point on the polar axis where the Mises stress is zero. In the accretion solution (b), the Mises stress is independent of polar angle β and reaches a maximum of 9.1 MPa at the center of the sphere.

4.3. Non-constant rotational speed

Eqs. (48)–(52) can be used to determine the stresses in bodies for which the rotational speed does not remain constant during accretion. We briefly consider the special case where the rotational speed increases in proportion to the square of the sphere's instantaneous radius $s(t)$,

$$\Omega = Cs^2; \quad C = \Omega_{\mathrm{f}}/a^2, \tag{72}$$

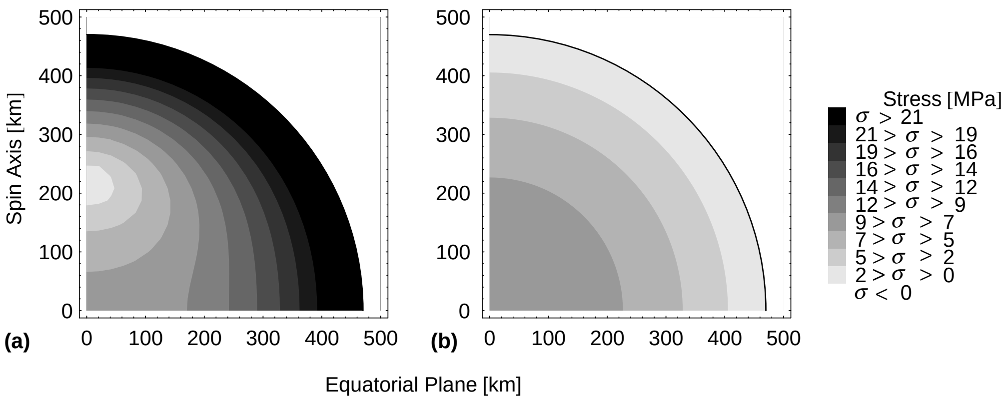

Fig. 1. Contour plot of the Mises stress for Ceres ($\rho = 2050$ kg/m^3, $\nu = 0.3$, $a = 470$ km, $\Omega = 0.0002$ rad/s), (a) using Chree's solution (no accretion) and (b) assuming accretion at constant speed.

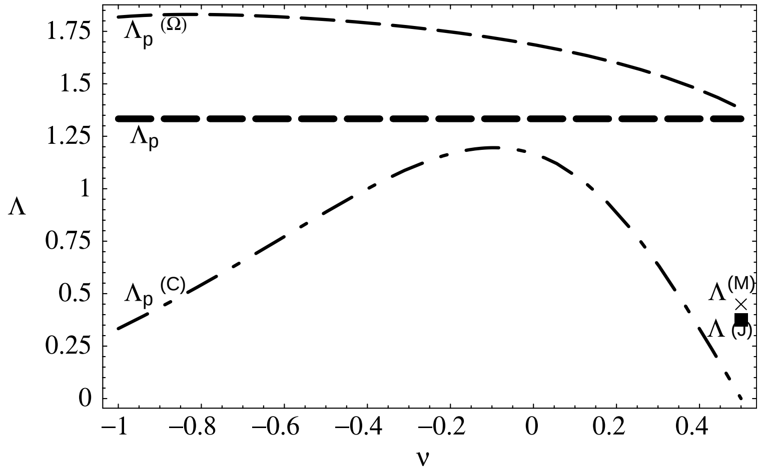

Fig. 2. Comparison of the failure limits, Λ, as a function of material compressibility as indicated by Poisson's ratio, ν. $\Lambda_P^{(C)}$: no accretion or residual stress, Λ_P: accretion with constant rotation rate, $\Lambda_P^{(\dot{\Omega})}$: accretion with quadratically increasing rotation rate. Incompressible finite kinematic fluid limits: $\Lambda^{(M)}$ MaClaurin spheroid, $\Lambda^{(J)}$ Jacobi ellipsoid.

such that when $s = a$, Ω equals the final rotational speed Ω_f. Dones and Tremaine (1993) suggest that this is an appropriate assumption for a highly dispersed cloud of particles under the gravitational influence of a large central body. The resulting stress field is found by substituting (72) into (48)–(52) and performing the necessary integrations.

It can be shown that the stress field is everywhere compressive if

$$\frac{\Omega_f^2}{\pi\rho G} \leqslant \frac{20(7+5\nu)(1-\nu)}{83-6\nu-45\nu^2} = \Lambda_P^{(\dot{\Omega})}. \tag{73}$$

Thus, when the rotational speed is not constant during accretion, we recover a dependence on material compressibility. Comparison of the maximum Mises stress and the maximum shear stress show that both are 9.5–11.5% lower than in the constant speed case with $\Omega = \Omega_f$.

4.4. Comparison of failure mechanisms

The failure limits predicted above can be compared as a function of material properties and with historical limits. Fig. 2 shows the limits identified in Eqs. (60), (71) and (73). Points above a particular curve denote rotation rates that would cause tensile stresses in a solid sphere according to the corresponding accretion scenario. Also shown in Fig. 2 are the classical incompressible fluid limits of Maclaurin spheroids and Jacobi ellipsoids (Chandrasekhar, 1969). This analysis indicates that solid spheres usually have greater rotational capability than incompressible fluids and that accreted and unaccreted spheres can have comparable or very different rotation limits depending upon the accretion scenario and the material constituencies.

5. Conclusions

A closed form solution has been found for the stress field in a self-gravitating elastic sphere that grows by an accretion process during which the rotational speed can be a general function of time. For accretion at constant rotational speed, the resulting principal stresses are everywhere compressive as long as the rotational speed satisfies a simple inequality (60).

The resulting stress field exhibits significant qualitative differences from the classical solution due to Chree (1895) for an initially unloaded and stress-free sphere loaded by gravitational and rotational inertia forces. In particular, the maximum shear stress and Mises stress for the accreting body depend only upon the rotational loading, whereas these stresses are dominated by gravitational effects in Chree's solution, except in the limiting case of incompressible materials.

Even though there are qualitative differences in the internal fields of accreted and unaccreted spheres, their rotation limits can still be comparable and depend critically upon the accretion scenario and material constitutive properties.

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